Generalized qubits of the vibrational motion of a trapped ion

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We present a method to generate qubits of the vibrational motion of an ion. The method is developed in the non-rotating wave approximation regime, therefore we consider regimes where the dynamics has not been studied. Because the solutions are valid for a more extended range of parameters we call them generalized qubits.

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Nonclassical states of the center-of-mass motion of a trapped ion have played an important role because of the potential practical applications such as precision spectroscopy [1] quantum computation [2, 3] and because of fundamental problems in quantum mechanics. Ways of producing Scrhödinger cat states [4], squeezed states [5], nonlinear coherent states [6, 7], number states, specific superpositions of them, and in particular, robust to noise (spontaneous emmision) qubits have been proposed [8]. In theoretical and experimental studies of a laser interacting with a single trapped ion it has been usually considered the case in which it may be modeled as a Jaynes-Cummings interaction [5, 9, 10], then exhibiting the peculiar features of this model like collapses and revivals [11], and the generation of nonclassical states common to such a model or (multi-photon) generalizations of it [12–14]. In treating this system usually two rotating wave approximations (RWA's) are done (the first related to the laser [optical] frequency and the second to the vibrational frequency of the ion), to remove counter-propagating terms of the Hamiltonian (that can not be treated analytically). Approximations on the Lamb-Dickke parameter, η, are usually done, considering it much smaller than unity. Additionaly, other approximations are done, based on the intensity of the laser shining on the trapped ion: the low-excitation regime $\Omega \ll \nu$ and the strong-excitation regime $\Omega \gg \nu$ [15], with Ω being the intensity of the field, and ν the vibrational frequency of the ion.

Recently there has been an alternative approach to the study of this dynamics [16]. In this approach a unitary transformation is used in order to linearise the ion-laser Hamiltonian. This transformation has been also used to propose schemes for realising faster logic gates for quantum information processing [3]. Under this unitary transformation the Hamiltonian takes exactly the form (note that *not* an effective form) of the Jaynes-Cummings Hamiltonian plus an extra term (an atomic driving term). In such a case a RWA may be done [16] in order to have an analytical solution for this problem, but it brings with it a new condition: Ω of the order of 2ν (note however that this is a new regime) and η still much less than unity. Later, another transformation [17] was used to diagonalize the linearized ion-laser Hamiltonian, without further conditions on Ω or η. This allowed the diagonalization of the Hamiltonian only in the ion basis. Exact eigenstates of the ion-laser Hamiltonian, i.e. trapping states for this system have been found [18], but because they do not form a basis, a complete (exact) solution may be found only for such states (eigenstates).

In this contribution we consider the complete Hamiltonian for the ion-laser interaction, linearise it as in [16] and further unitarily transform it, without performing the RWA to obtain an effective Hamiltonian that can be easily solved . This is an extension of a method of small rotations recently applied by Klimov and Sánchez-Soto [19] to the Dicke model and other systems (they apply small rotations on the atomic basis). This allows us to obtain a more general solution valid in a more extended range of parameters. We apply this solution to a simple initial state and show that qubits may be produced. Because may be produced with less constrains on the parameters, we call them generalized qubits.

We consider the Hamiltonian that describes the interaction of a single two-level trapped ion with a laser beam (we set $\hbar = 1$) [9, 15]

$$\hat{H} = \nu\hat{n} + \frac{\delta}{2}\hat{\sigma}_z + \Omega(\hat{\sigma}_- e^{-i\eta(\hat{a}+\hat{a}^\dagger)} + \hat{\sigma}_+ e^{i\eta(\hat{a}+\hat{a}^\dagger)}), \quad (1)$$

where $\hat{n} = \hat{a}^\dagger\hat{a}$, with $\hat{a}^\dagger$ ($\hat{a}$) the ion vibratonial creation (anihilation) operator, and $\hat{\sigma}_+ = |e\rangle\langle g|$ ($\hat{\sigma}_- = |g\rangle\langle e|$) is the electronic raising (lowering) operator, $|e\rangle$ ($|g\rangle$) denoting the excited (ground) state of the ion. The detuning δ is defined as the difference between the atomic transition frequency (ω_0) and the frequency of laser(ω_L).

Applying the unitary transformation $\hat{T}_1$ [16]

$$\hat{T}_1 = \frac{1}{\sqrt{2}}\left(\frac{1}{2}[\hat{D}^\dagger(\beta) + \hat{D}(\beta)]\hat{I} + \frac{1}{2}[\hat{D}^\dagger(\beta) - \hat{D}(\beta)]\hat{\sigma}_z - \hat{D}^\dagger(\beta)\sigma_- + \hat{D}(\beta)\sigma_+\right), \quad (2)$$

(where $\hat{I} = |g\rangle\langle g| + |e\rangle\langle e|$ and $\hat{D}(\beta)$ is the displacement operator with $\beta = \frac{i\eta}{2}$ its amplitude) to (1), we obtain the linearised Hamiltonian (in the on resonance case i. e. $\delta = 0$), given by [16]

$$\hat{H}_1 = \hat{T}_1 \hat{H} \hat{T}_1^\dagger = \nu\hat{n} + \Omega\hat{\sigma}_z + i\frac{\eta\nu}{2}(\hat{a} - \hat{a}^\dagger)(\hat{\sigma}_- + \hat{\sigma}_+) + \frac{\nu\eta^2}{4}. \tag{3}$$

We now apply the unitary tranformation

$$\hat{T}_2 = e^{-i\epsilon(\hat{a}+\hat{a}^\dagger)(\hat{\sigma}_+ + \hat{\sigma}_-)}, \tag{4}$$

to (3) to obtain $\hat{H}_2 = \hat{T}_2 \hat{H}_1 \hat{T}_2^\dagger$,

$$\begin{aligned}\hat{H}_2 &= \nu\left(\hat{n} + i\epsilon(\hat{a} - \hat{a}^\dagger)(\hat{\sigma}_- + \hat{\sigma}_+) + \epsilon^2\right) \\ &\quad + \Omega\left(\sigma_z \cos[2\epsilon(\hat{a} + \hat{a}^\dagger)] + i(\sigma_- - \sigma_+)\sin[2\epsilon(\hat{a} + \hat{a}^\dagger)]\right) \\ &\quad + i\frac{\eta\nu}{2}(\hat{a} - \hat{a}^\dagger)(\hat{\sigma}_- + \hat{\sigma}_+) + \frac{\nu\epsilon\eta^2}{4}.\end{aligned} \tag{5}$$

By considering $\epsilon \ll 1$, we can approximate (5) as (we disregard constant terms that only contribute to a shift of energies)

$$\begin{aligned}\hat{H}_2 &\approx \nu\left(\hat{n} + i\epsilon(\hat{a} - \hat{a}^\dagger)(\hat{\sigma}_- + \hat{\sigma}_+)\right) + \Omega\left(\sigma_z + 2i\epsilon(\sigma_- - \sigma_+)(\hat{a} + \hat{a}^\dagger)\right) \\ &\quad + i\frac{\eta\nu}{2}(\hat{a} - \hat{a}^\dagger)(\hat{\sigma}_- + \hat{\sigma}_+),\end{aligned} \tag{6}$$

and by setting

$$\epsilon = -\frac{\eta}{2}\frac{\nu}{\nu + 2\Omega}, \tag{7}$$

we finally obtain

$$\hat{H}_2 = \nu\hat{n} + \Omega\hat{\sigma}_z + i\lambda(\hat{\sigma}_+\hat{a} - \hat{a}^\dagger\hat{\sigma}_-). \tag{8}$$

Note that the *coupling constant*, $\lambda = \frac{2\eta\nu\Omega}{\nu + 2\Omega}$, has changed (before it was $\frac{\eta\nu}{2}$).

We should remark that transformation (4) requires $\epsilon \ll 1$ and this may be achieved in different forms:

a) $\eta \ll 1$ and ν and Ω *any* numbers (this is, it can be $\nu \ll \Omega$, $\nu \gg \Omega$ or of the same order of magnitude);

b) no restrictions on η and $\nu \ll \Omega$ or,

c) $\eta < 1$ and $\nu < \Omega$ (note that, for instance, a value of $\eta = 0.3$ and $\Omega = 2\nu$ gives $\epsilon = -0.03$).

The three possibilities above allow the Hamiltonian to be approximated to first order and to disregard terms of second and higher orders.

We are now in the position to give a solution to the Hamiltonian (1), that reads

$$|\Psi(t)\rangle = \hat{T}^\dagger \hat{U} \hat{T} |\Psi(0)\rangle, \tag{9}$$

where we have written

$$\begin{aligned}\hat{T} &= \hat{T}_2\hat{T}_1 \\ &= \frac{1}{\sqrt{2}}\left(\frac{1}{2}[\hat{D}^\dagger(\beta_-) + \hat{D}(\beta_-)]\hat{I} + \frac{1}{2}[\hat{D}^\dagger(\beta_-) - \hat{D}(\beta_-)]\hat{\sigma}_z - \hat{D}^\dagger(\beta_-)\sigma_- + \hat{D}(\beta_-)\sigma_+\right),\end{aligned} \tag{10}$$

with $\beta_- = i(\frac{\eta}{2} - \epsilon)$ and where $|\Psi(0)\rangle$ is the initial wave function and $\hat{U}$ is the evolution operator of the off-resonant Jaynes-Cummings Model

$$\hat{U} = e^{-it(\nu\hat{n} + \frac{1}{2}\nu\hat{\sigma}_z)} e^{-it[\Delta\hat{\sigma}_z + i\lambda(\hat{a}\hat{\sigma}_+ - \hat{a}^\dagger\hat{\sigma}_-)]}, \tag{11}$$

where we defined $\Delta = \Omega - \frac{\nu}{2}$. Equation (11) may be finally written in the form [20]

$$\hat{U} = e^{-it(\nu\hat{n} + \frac{1}{2}\nu\hat{\sigma}_z)}\left(\frac{1}{2}[\hat{U}_{11} + \hat{U}_{22}]\hat{I} + \frac{1}{2}[\hat{U}_{11} - \hat{U}_{22}]\hat{\sigma}_z + \hat{U}_{21}\sigma_- + \hat{U}_{12}\sigma_+\right), \tag{12}$$

with

$$\hat{U}_{11} = \cos\hat{\alpha}_{\hat{n}+1}t - i\Delta\frac{\sin\hat{\alpha}_{\hat{n}+1}t}{\hat{\alpha}_{\hat{n}+1}}, \tag{13}$$

$$\hat{U}_{12} = \lambda\hat{a}\frac{\sin\hat{\alpha}_{\hat{n}}t}{\hat{\alpha}_{\hat{n}}}, \tag{14}$$

$$\hat{U}_{21} = -\lambda\hat{a}^{\dagger}\frac{\sin\hat{\alpha}_{\hat{n}+1}t}{\hat{\alpha}_{\hat{n}+1}}, \tag{15}$$

and

$$\hat{U}_{22} = \cos\hat{\alpha}_{\hat{n}}t + i\Delta\frac{\sin\hat{\alpha}_{\hat{n}}t}{\hat{\alpha}_{\hat{n}}}, \tag{16}$$

where

$$\hat{\alpha}_{\hat{n}} = \sqrt{\Delta^2 + \lambda^2\hat{n}}. \tag{17}$$

By considering the ion initially in its excited state and in a coherent (vibrational) state with amplitude β_-, i.e.

$$|\Psi(0)\rangle = |e\rangle|\beta_-\rangle, \tag{18}$$

we obtain the evolved wave function at time t

$$\begin{aligned} |\Psi(t)\rangle &= \frac{1}{2}\hat{D}(\beta_-)\left([e^{-i\frac{\nu t}{2}}(\cos\alpha_1 t - i\frac{\Delta}{\alpha_1}\sin\alpha_1 t) + e^{i\Omega t}]|0\rangle + \frac{\lambda}{\alpha_1}e^{-i\frac{\nu t}{2}}\sin\alpha_1 t|1\rangle\right)|e\rangle \\ &+ \frac{1}{2}\hat{D}^{\dagger}(\beta_-)\left([e^{-i\frac{\nu t}{2}}(\cos\alpha_1 t - i\frac{\Delta}{\alpha_1}\sin\alpha_1 t) - e^{i\Omega t}]|0\rangle - \frac{\lambda}{\alpha_1}e^{-i\frac{\nu t}{2}}\sin\alpha_1 t|1\rangle\right)|g\rangle \end{aligned} \tag{19}$$

with $\alpha_1 = \langle 1|\hat{\alpha}_{\hat{n}}|1\rangle$. Note that $\hat{D}(\beta_-)|k\rangle = |\beta_-, k\rangle$ is a displaced number state [21]. By measuring either the excited state or ground state of the ion (there are standard techniques to do so, see for instance [9]), we end up with a superposition of displaced number states with amplitude β_- or $-\beta_-$, respectively. Therefore, by displacing the resulting state if the ion is measured in the excited state by $-\beta_-$ (or by β_- if measured in the ground state, we produce a qubit of the vibrational motion of an ion. To clarify this point, let us assume the ion is measured in the excited state. The vibrational wave function collapses to the state

$$|\Psi_{vib}(t)\rangle = \frac{1}{N}\hat{D}(\beta_-)\left([e^{-i\frac{\nu t}{2}}(\cos\alpha_1 t - i\frac{\Delta}{\alpha_1}\sin\alpha_1 t) + e^{i\Omega t}]|0\rangle + \frac{\lambda}{\alpha_1}e^{-i\frac{\nu t}{2}}\sin\alpha_1 t|1\rangle\right), \tag{20}$$

where N is the normalization constant. By finally displacing the state by an amplitude $-\beta_-$, we obtain the (qubit) state

$$|\Psi_d(t)\rangle = \frac{1}{N}\left([e^{-i\frac{\nu t}{2}}(\cos\alpha_1 t - i\frac{\Delta}{\alpha_1}\sin\alpha_1 t) + e^{i\Omega t}]|0\rangle + \frac{\lambda}{\alpha_1}e^{-i\frac{\nu t}{2}}\sin\alpha_1 t|1\rangle\right). \tag{21}$$

Moreover, without (conditional) measurement, and by controling the interaction time between the ion and the laser beam, setting it to $\alpha_1 t = \pi$, the (Schrödinger cat) state

$$|\Psi(\alpha_1 t = \pi)\rangle = \frac{1}{2}\left((e^{i\Omega t} - e^{-i\nu t})|\beta_-\rangle|e\rangle - (e^{i\Omega t} + e^{-i\nu t})|-\beta_-\rangle|g\rangle\right), \tag{22}$$

is generated. This state was produced in [4] and also studied in [22]. Note that the quantity β_- is slightly larger than the Lamb-Dicke parameter, which in general is not needed to be small.

In this contribution we have suceeded in solving the problem of trapped ion interacting resonantly ($\delta = 0$ case) with a laser field whitout making the rotating wave aproximation, but applying a method of small rotations that allowed such solution. It should be remarked that now there are new regimes that can be reached, since the usual approximations as the low intensity or high intensity regimes were not considered.

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